

# Minimality in topological groups and Heisenberg type groups

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## Abstract

We study relatively minimal subgroups in topological groups. We find, in particular, some natural relatively minimal subgroups in unipotent groups which are defined over "good" rings. By "good" rings we mean archimedean absolute valued (not necessarily associative) division rings. Some of the classical rings which we consider besides the field of reals are the ring of quaternions and the ring of octonions. This way we generalize in part a previous result which was obtained by Dikranjan and Megrelishvili [1] and involved the Heisenberg group.

## 1 Introduction

A Hausdorff topological group  $G$  is minimal if  $G$  does not admit a strictly coarser Hausdorff group topology or equivalently if every injective continuous group homomorphism  $G \rightarrow P$  into a Hausdorff topological group is a topological embedding. The concept of minimal topological groups was introduced by Stephenson [9] and Doichinov [2] in 1971 as a natural generalization of compact groups.

Heisenberg group and more precisely its generalization, which we present in section 2 (see also [4, 7]), provides many examples of minimal groups.

Recently Dikranjan and Megrelishvili [1] introduced the concept of *co-minimality* (see Definition 2.5) of subgroups in topological groups after the latter author had introduced the concept of *relative minimality* (see Definition 2.3 and also [3]) of subgroups in topological groups and found such subgroups in a *generalized Heisenberg group* (see [4, 7]).

In [1, Proposition 2.4.2] Megrelishvili and Dikranjan proved that the canonical bilinear mapping  $V \times V^* \rightarrow \mathbb{R}$ ,  $\langle v, f \rangle = f(v)$  is *strongly minimal* (see Definition 2.7) for all normed spaces  $V$ .

The following result is obtained as a particular case: The inner product map

$$\mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$$

is strongly minimal. The latter result leads in [1] and [3] to the conclusion that for every  $n \in \mathbb{N}$  the subgroups

$$\left\{ \begin{pmatrix} 1 & a & 0 \\ 0 & I_n & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid a \in \mathbb{R}^n \right\}, \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \mid b \in \mathbb{R}^n \right\}$$

are relatively minimal in the group

$$\left\{ \begin{pmatrix} 1 & a & c \\ 0 & I_n & b \\ 0 & 0 & 1 \end{pmatrix} \middle| a, b \in \mathbb{R}^n, c \in \mathbb{R} \right\}$$

which is known as the classical  $2n + 1$ -dimensional Heisenberg group (where  $I_n$  denotes the identity matrix of size  $n$ ). Theorem 3.4 and Corollary 3.6 generalize these results and allow us to replace the field of reals by every other archimedean absolute valued (not necessarily associative) division ring, for example, they can be applied for the ring of quaternions and the ring of octonions. Theorem 3.9 provides a different generalization. It generalizes the case of the classical real 3-dimensional Heisenberg group. We consider for every  $n \in \mathbb{N}$  the group of upper unitriangular matrices over an archimedean absolute valued field of size  $n + 2 \times n + 2$  and we find relatively minimal subgroups of this group. This result is a generalization since the classical real 3-dimensional Heisenberg group is a unitriangular group. This theorem is not new when we take  $n = 1$  and consider the field to reals. However, we obtain a new result even for  $\mathbb{R}$  when we take  $n > 1$ . This theorem can also be applied for the fields  $\mathbb{Q}$  and  $\mathbb{C}$ .

## 2 Minimality in Generalized Heisenberg groups

The group

$$H = \left\{ \begin{pmatrix} 1 & x & a \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \middle| x, y, a \in \mathbb{R} \right\} \cong (\mathbb{R} \times \mathbb{R}) \rtimes \mathbb{R}$$

is known as the classical real 3-dimensional *Heisenberg Group*.

We need a far reaching generalization [4, 7, 3], the *generalized Heisenberg group*, which is based on biadditive mappings.

**Definition 2.1** *Let  $E, F, A$  be abelian groups. A map  $w : E \times F \rightarrow A$  is said to be biadditive if the induced mappings*

$$w_x : F \rightarrow A, w_f : E \rightarrow A, w_x(f) := w(x, f) =: w_f(x)$$

*are homomorphisms for all  $x \in E$  and  $f \in F$ .*

**Definition 2.2** [3, Definition 1.1] *Let  $E, F$  and  $A$  be Hausdorff abelian topological groups and  $w : E \times F \rightarrow A$  be a continuous biadditive mapping. Denote by  $H(w) = (A \times E) \rtimes F$  the topological semidirect product (say, generalized Heisenberg group induced by  $w$ ) of  $F$  and the group  $A \times E$ . The group operation is defined as follows: for a pair*

$$u_1 = (a_1, x_1, f_1), u_2 = (a_2, x_2, f_2)$$

*we define*

$$u_1 u_2 = (a_1 + a_2 + f_1(x_2), x_1 + x_2, f_1 + f_2)$$

*where,  $f_1(x_2) = w(x_2, f_1)$ . Then  $H(w)$  becomes a Hausdorff topological group. In the case of a normed space  $X$  and a canonical biadditive function  $w : X \times X^* \rightarrow \mathbb{R}$   $(x, f) \mapsto f(x)$*

(where  $X^*$  is the Banach space of all continuous functionals from  $X$  to  $\mathbb{R}$ , known as the dual space of  $X$ ) we write  $H(X)$  instead of  $H(w)$ .

**Definition 2.3** [1, Definition 1.1.1] Let  $X$  be a subset of a Hausdorff topological group  $(G, \tau)$ . We say that  $X$  is relatively minimal in  $G$  if every coarser Hausdorff group topology  $\sigma \subset \tau$  of  $G$  induces on  $X$  the original topology. That is,  $\sigma|_X = \tau|_X$ .

**Theorem 2.4** [3, Theorem 2.2] The subgroups  $X$  and  $X^*$  are relatively minimal in the generalized Heisenberg group  $H(X) = (\mathbb{R} \times X) \rtimes X^*$  for every normed space  $X$ .

The concept of co-minimality which is presented below played a major role in generalizing and strengthen Theorem 2.4. Let  $H$  be a subgroup of a topological group  $(G, \gamma)$ . The quotient topology on the left coset space  $G/H := \{gH\}_{g \in G}$  will be denoted by  $\gamma/H$ .

**Definition 2.5** [1, Definition 1.1.2] Let  $X$  be a topological subgroup of a Hausdorff topological group  $(G, \tau)$ . We say that  $X$  is co-minimal in  $G$  if every coarser Hausdorff group topology  $\sigma \subset \tau$  of  $G$  induces on the coset space  $G/X$  the original topology. That is,  $\sigma/X = \tau/X$ .

**Definition 2.6** Let  $E, F, A$  be abelian Hausdorff groups. A biadditive mapping  $w : E \times F \rightarrow A$  will be called separated if for every pair  $(x_0, f_0)$  of nonzero elements there exists a pair  $(x, f)$  such that  $f(x_0) \neq 0_A$  and  $f_0(x) \neq 0_A$ , where  $f(x) = w(x, f)$ .

**Definition 2.7** [1, Definition 2.2] Let  $(E, \sigma), (F, \tau), (A, \nu)$  be abelian Hausdorff topological groups. A continuous separated biadditive mapping

$$w : (E, \sigma) \times (F, \tau) \rightarrow (A, \nu)$$

will be called strongly minimal if for every coarser triple  $(\sigma_1, \tau_1, \nu_1)$  of Hausdorff group topologies  $\sigma_1 \subset \sigma, \tau_1 \subset \tau, \nu_1 \subset \nu$  such that

$$w : (E, \sigma_1) \times (F, \tau_1) \rightarrow (A, \nu_1)$$

is continuous (in such cases we say that the triple  $(\sigma_1, \tau_1, \nu_1)$  is compatible) it follows that  $\sigma_1 = \sigma, \tau_1 = \tau$ . We say that the biadditive mapping is minimal if  $\sigma_1 = \sigma, \tau_1 = \tau$  holds for every compatible triple  $(\sigma_1, \tau_1, \nu)$  (with  $\nu_1 := \nu$ ).

**Remark 2.8** The multiplication map  $A \times A \rightarrow A$  is minimal for every Hausdorff topological unital ring  $A$ . However note that the multiplication map  $\mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z}$  (being minimal) is not strongly minimal.

The following theorem which uses the concept of co-minimality and strongly biadditive mappings generalizes Theorem 2.4.

**Theorem 2.9** [1, Theorem 4.1] Let  $w : (E, \sigma) \times (F, \tau) \rightarrow (A, \nu)$  be a strongly minimal biadditive mapping. Then:

1.  $A, A \times E$  and  $A \times F$  are co-minimal subgroups of the Heisenberg group  $H(w)$ .
2.  $E \times F$  is a relatively minimal subset in  $H(w)$ .
3. The subgroups  $E$  and  $F$  are relatively minimal in  $H(w)$ .

**Remark 2.10** *The mapping  $w : X \times X^* \rightarrow \mathbb{R} \ (x, f) \mapsto f(x)$  is strongly minimal for every normed space  $X$ . Therefore, Theorem 2.9 is indeed a generalization of Theorem 2.4.*

**Corollary 2.11** *[1, Corollary 4.2] The following conditions are equivalent:*

1.  $H(w)$  is a minimal group.
2.  $w$  is a minimal biadditive mapping and  $A$  is a minimal group.

Since  $\mathbb{Z}$  with the  $p$ -adic topology  $\tau_p$  is a minimal group for every prime  $p$  [6] the following corollary is obtained by Remark 2.8:

**Corollary 2.12** *[1, Corollary 4.6.2] The Heisenberg group  $H(w) = (\mathbb{Z} \times \mathbb{Z}) \rtimes \mathbb{Z}$  of the mapping  $(\mathbb{Z}, \tau_p) \times (\mathbb{Z}, \tau_p) \rightarrow (\mathbb{Z}, \tau_p)$  is a minimal two step nilpotent precompact group for every  $p$ -adic topology  $\tau_p$ .*

### 3 Topological rings and absolute values

In this paper rings are not assumed to be necessarily associative. However, when we consider division rings we assume they are associative unless otherwise is stated.

**Definition 3.1** *An absolute value  $A$  on a (not necessarily associative) division ring  $K$  is archimedean if there exists  $n \in \mathbb{N}$  such that  $A(n) > 1$  (where, for any  $n \in \mathbb{N}$ ,  $n := n.1 = 1 + \dots + 1$  ( $n$  terms)).*

From now on we use the following notations for a commutative group  $G$  which is denoted additively: the zero element is denoted by  $0_G$ . If  $G$  is also a ring with multiplicative unit we denote this element by  $1_G$ . In the case of a group  $G$  which is a direct product of groups we shall use slightly different notation and denote the zero element by  $\bar{0}_G$ .

**Lemma 3.2** *Let  $X$  be a (not necessarily associative) division ring with an archimedean absolute value  $A$  and denote by  $\tau$  the ring topology induced by the absolute value. Let  $\sigma \subset \tau$  be a strictly coarser group topology with respect to the additive structure of  $X$ . Then, every  $\sigma$ -neighborhood of  $0_X$  is unbounded with respect to the absolute value.*

*Proof.* Since  $\sigma$  is strictly coarser than  $\tau$ , there exists an open ball  $B(0, r)$  with  $r > 0$  not containing any  $\sigma$ -neighborhood of  $0_X$ . Then, for every  $\sigma$ -neighborhood  $U$  of  $0_X$  there exists  $x$  in  $U$  such that  $A(x) \geq r$ . Fix a  $\sigma$ -neighborhood  $V$  of  $0_X$ . We show that  $V$  is unbounded with respect to the absolute value  $A$ . Since  $A$  is an archimedean absolute value there exists  $n_0 \in \mathbb{N}$  such that  $A(n_0) = c > 1$ . Clearly, for every  $m \in \mathbb{N}$  there exists a  $\sigma$ -neighborhood  $W$  of  $0_X$  such that

$$\underbrace{W + W + \dots + W}_{n_0^m} \subset V.$$

By our assumption there exists  $x \in W$  such that  $A(x) \geq r$ . Now for the element

$$n_0^m x := \underbrace{x + x + \cdots + x}_{n_0^m} \in V$$

we obtain that  $A(n_0^m x) = A(n_0)^m A(x) \geq c^m r$ . This clearly means that  $V$  is unbounded.  $\square$

**Lemma 3.3** *Let  $(G_i)_{i \in I}$  be a family of topological groups. For each  $i \in I$  denote by  $\tau_i$  the topology of  $G_i$  and by  $p_i$  the projection of  $G := \prod_{i \in I} G_i$  to  $G_i$ . Suppose that  $\sigma$  is a group topology on  $G$  which is strictly coarser than the product topology on  $G$  denoted by  $\tau$ . Then there exist  $j \in I$  and a group topology  $\sigma_j$  on  $G_j$  which is strictly coarser than  $\tau_j$ , such that  $\mathfrak{B}_j = p_j(\mathfrak{B})$ , where  $\mathfrak{B}_j$  is the neighborhood filter of  $0_{G_j}$  with respect to  $\sigma_j$  and  $\mathfrak{B}$  is the neighborhood filter of  $\bar{0}_G$  with respect to  $\sigma$ .*

*Proof.* Since the topology  $\sigma$  is strictly coarser than  $\tau$  which is the product topology on  $G$ , we get that there exists  $j \in I$  for which the projection  $p_j : (G, \sigma) \rightarrow (G_j, \tau_j)$  is not continuous at  $\bar{0}_G$ . Hence, there exist a  $\tau_j$ -neighborhood  $V$  of  $0_{G_j}$  such that  $p_j(O) \not\subseteq V$  for every  $O \in \mathfrak{B}$ . Hence, if  $p_j(\mathfrak{B})$  is the neighborhood filter of  $0_{G_j}$  for some group topology  $\sigma_j$  on  $G_j$  then this topology is strictly coarser than  $\tau_j$ . We shall prove that this formulation defines a group topology  $\sigma_j$ . Indeed, consider the normal subgroup  $H = \prod_{i \in I} F_i$  of  $G$  where

$$F_i = \begin{cases} G_i & \text{if } i \neq j \\ \{0_{G_i}\} & \text{if } i = j \end{cases}.$$

It is easy to show that  $(G_j, \tau_j)$  is topologically isomorphic to the quotient group  $G/H$  of  $(G, \tau)$ . Let  $\sigma_j$  be the finest topology on  $G_j$  for which the projection  $p_j : (G, \sigma) \rightarrow G_j$  is continuous. It is exactly the quotient topology on  $G_j = G/H$  for the topological group  $(G, \sigma)$ . By our construction  $\sigma_j$  is strictly coarser than  $\tau_j$ . Then indeed  $\sigma_j$  is the desired group topology on  $G_j$  and  $\mathfrak{B}_j = p_j(\mathfrak{B})$  is the desired neighborhood filter.  $\square$

**Theorem 3.4** *Let  $F$  be a (not necessarily associative) division ring furnished with an archimedean absolute value  $A$ . For each  $n \in \mathbb{N}$ ,*

$$w_n : F^n \times F^n \mapsto F, \quad w_n(\bar{x}, \bar{y}) = \sum_{i=1}^n x_i y_i$$

*(where  $(\bar{x}, \bar{y}) = ((x_1, \dots, x_n), (y_1, \dots, y_n))$  is a strongly minimal biadditive mapping.*

*Proof.* Clearly, for each  $n \in \mathbb{N}$ ,  $w_n$  is a continuous separated biadditive mapping. Denote by  $\tau$  the topology of  $F$  induced by  $A$  and by  $\tau^n$  the product topology on  $F^n$ . Consider the max-metric  $d$  on  $F^n$ . Then its topology is exactly  $\tau^n$ . Let  $(\sigma, \sigma', \nu)$  be a compatible triple with respect to  $w_n$ . We prove that  $\sigma = \sigma' = \tau^n$ . Assuming the contrary we get that at least one of the group topologies  $\sigma, \sigma'$  is strictly coarser than  $\tau^n$ . We first assume that  $\sigma$  is strictly coarser than  $\tau^n$ . Since  $\nu$  is Hausdorff and  $(\sigma, \sigma', \nu)$  is compatible there exist a  $\nu$ -neighborhood  $Y$  of  $0 := 0_F$  and  $V, W$  which are respectively  $\sigma, \sigma'$ -neighborhoods of  $\bar{0}_{F^n}$  such that  $VW \subset Y$  and in addition  $1_F \notin Y$ . Since  $W \in \sigma' \subset \tau^n$ , then there exists  $\epsilon_0 > 0$  such that the corresponding  $d$ -ball  $B(0, \epsilon_0)$  is a subset of  $W$ . Since  $\sigma$  is

strictly coarser than  $\tau^n$  (by Lemmas 3.2 and 3.3) there exists  $i \in I := \{1, 2, \dots, n\}$  such that  $p_i(V)$  is norm unbounded. Therefore, there exists  $\bar{x} \in V$  such that  $A(p_i(\bar{x})) > \frac{1}{\epsilon_0}$ . Hence,  $A((p_i(\bar{x}))^{-1}) < \epsilon_0$ . Now, let us consider a vector  $\bar{a} \in F^n$  such that for every  $j \neq i$ ,  $a_j = 0$  and  $a_i = (p_i(\bar{x}))^{-1}$ . Clearly,  $\bar{a} \in B(0, \epsilon_0) \subset W$ . We then get that  $w_n(\bar{x}, \bar{a}) = 1_F \in VW \subset Y$ . This contradicts our assumption. Using the same technique we can show that  $\sigma'$  can't be strictly coarser than  $\tau^n$ .  $\square$

**Example 3.5** 1. Let  $F \in \{\mathbb{Q}, \mathbb{R}, \mathbb{C}\}$  with the usual absolute value. Then for each  $n \in \mathbb{N}$  the map

$$w_n : F^n \times F^n \mapsto F$$

is strongly minimal. The case of  $F$  equals to  $\mathbb{R}$  follows also from [1, Proposition 2.42].

2. For each  $n \in \mathbb{N}$  the map

$$w_n : \mathbb{H}^n \times \mathbb{H}^n \mapsto \mathbb{H}$$

is strongly minimal, where  $\mathbb{H}$  is the quaternions ring equipped with the archimedean absolute value defined by:

$$\|q\| = (a^2 + b^2 + c^2 + d^2)^{\frac{1}{2}}$$

for each  $q = a + bi + cj + dk \in \mathbb{H}$ .

3. Let  $G$  be the non-associative ring of octonions.

This ring can be defined (see [11]) as pairs of quaternions (this is the Cayley-Dickson construction). Addition is defined pairwise. The product of two pairs of quaternions  $(a, b)$  and  $(c, d)$  is defined by  $(a, b)(c, d) = (ac - db^*, a^*d + cb)$  where  $z^* = e - fi - gj - hk$  denotes the conjugate of  $z = e + fi + gj + hk$ . We define a norm on  $G$  as follows:

$$\|(a + bi + cj + dk, e + fi + gj + hk)\| = (a^2 + b^2 + c^2 + d^2 + e^2 + f^2 + g^2 + h^2)^{\frac{1}{2}}.$$

This norm agrees with the standard Euclidean norm on  $\mathbb{R}^8$ . It can be proved that for each  $x_1, x_2 \in G$ ,  $\|x_1 x_2\| = \|x_1\| \cdot \|x_2\|$  hence  $\|\cdot\|$  is an absolute value and clearly it is archimedean. Again by Theorem 3.4 the map

$$w_n : G^n \times G^n \mapsto G$$

is strongly minimal for each  $n \in \mathbb{N}$ .

**Corollary 3.6** Under the conditions of Theorem 3.4 we obtain the following results:

1.  $(F \times \{\bar{0}_{F^n}\}) \curlywedge \{\bar{0}_{F^n}\}$ ,  $(F \times F^n) \curlywedge \{\bar{0}_{F^n}\}$  and  $(F \times \{\bar{0}_{F^n}\}) \curlywedge F^n$  are co-minimal subgroups of the Heisenberg group  $H(w_n)$ .
2.  $(\{0_F\} \times F^n) \curlywedge F^n$  is a relatively minimal subset in  $H(w_n)$ .
3. The subgroups  $(\{0_F\} \times F^n) \curlywedge \{\bar{0}_{F^n}\}$  and  $(\{0_F\} \times \{\bar{0}_{F^n}\}) \curlywedge F^n$  are relatively minimal in  $H(w_n)$ .

*Proof.* Apply Theorem 2.9 to the strongly minimal biadditive mapping  $w_n$ .  $\square$

**Remark 3.7** We replace  $H(w_n)$  by  $H(F^n)$  for convenience ( $w_n$  is the strongly minimal biadditive mapping from 3.4). In terms of matrices:  $H(F^n)$  is the  $2n + 1$ -dimensional Heisenberg group with coefficients from  $F$  which consists of square matrices of size  $n + 2$ :

$$A = \begin{pmatrix} 1_F & x_1 & x_2 & \dots & x_{n-1} & x_n & r \\ 0_F & 1_F & 0_F & 0_F & 0_F & 0_F & y_1 \\ 0_F & 0_F & \ddots & \ddots & \ddots & \vdots & y_2 \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0_F & \vdots \\ \vdots & \vdots & \ddots & \ddots & 1_F & 0_F & y_{n-1} \\ 0_F & 0_F & \ddots & \ddots & 0_F & 1_F & y_n \\ 0_F & 0_F & 0_F & \dots & \dots & 0_F & 1_F \end{pmatrix}$$

and by the result (2) of Corollary 3.6 we obtain that the set of matrices

$$B = \begin{pmatrix} 1_F & x_1 & x_2 & \dots & x_{n-1} & x_n & 0_F \\ 0_F & 1_F & 0_F & 0_F & 0_F & 0_F & y_1 \\ 0_F & 0_F & \ddots & \ddots & \ddots & \vdots & y_2 \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0_F & \vdots \\ \vdots & \vdots & \ddots & \ddots & 1_F & 0_F & y_{n-1} \\ 0_F & 0_F & \ddots & \ddots & 0_F & 1_F & y_n \\ 0_F & 0_F & 0_F & \dots & \dots & 0_F & 1_F \end{pmatrix}$$

is a relatively minimal subset of  $H(F^n)$ .

**Lemma 3.8** 1. If  $H$  is a subgroup of a topological group  $(G, \tau)$  and  $X$  is a relatively minimal subset in  $H$ , then  $X$  is also relatively minimal in  $G$ .

2. Let  $(G_1, \tau_1), (G_2, \tau_2)$  be topological groups and  $H_1, H_2$  be their subgroups (respectively). If  $H_1$  is relatively minimal in  $G_1$  and there exists a topological isomorphism  $f : (G_1, \tau_1) \rightarrow (G_2, \tau_2)$  such that the restriction to  $H_1$  is a topological isomorphism onto  $H_2$ , then  $H_2$  is relatively minimal in  $G_2$ .

3. Let  $(G, \tau)$  be a topological group and let  $X$  be a subset of  $G$ . If  $X$  is relatively minimal in  $(G, \tau)$ , then every subset of  $X$  is also relatively minimal in  $(G, \tau)$ .

*Proof.* (1): Let  $\sigma \subset \tau$  be a coarser Hausdorff group topology of  $G$ , then  $\sigma|_H \subset \tau|_H$  is a coarser Hausdorff group topology of  $H$ . Since  $X$  is a relatively minimal subset in  $H$ , we get that

$$\sigma|_X = (\sigma|_H)|_X = (\tau|_H)|_X = \tau|_X.$$

Hence,  $X$  is relatively minimal in  $G$ .

(2): Observe that if  $\sigma_2 \subset \tau_2$  is a coarser Hausdorff group topology of  $G_2$ , then

$$f^{-1}(\sigma_2) = \{f^{-1}(U) \mid U \in \sigma_2\} \subset \tau_1$$

is a coarser group topology of  $G_1$ . Since  $H_1$  is relatively minimal in  $(G_1, \tau_1)$  we obtain that  $\tau_1|_{H_1} = f^{-1}(\sigma_2)|_{H_1}$ . This implies that  $\tau_2|_{H_2} = \sigma_2|_{H_2}$ . This completes our proof.

(3): Let  $Y$  be a subset of  $X$  and  $\sigma \subset \tau$  a coarser Hausdorff group topology. Then, by the fact that  $X$  is relatively minimal in  $(G, \tau)$  and since  $Y$  is a subset of  $X$  we obtain that

$$\sigma|_Y = (\sigma|_X)|_Y = (\tau|_X)|_Y = \tau|_Y.$$

Hence,  $Y$  is relatively minimal in  $G$ .  $\square$

The following is new even for the case of  $F = \mathbb{R}$  (for  $n > 1$ ).

**Theorem 3.9** *Let  $F$  be a field furnished with an archimedean absolute value  $A$ . For all  $n \in \mathbb{N}$  denote by  $U_{n+2}(F)$  the topological group of all  $n+2 \times n+2$  upper unitriangular matrices with entries from  $F$ . Then  $\forall n \in \mathbb{N}$  and for each  $i, j$  such that  $i < j$ ,  $(i, j) \neq (1, n+2)$  each of the subgroups*

$$G_{ij}^{n+2}(F) := \left\{ \begin{pmatrix} 1_F & 0_F & 0_F & 0_F & 0_F \\ 0_F & 1_F & 0_F & 0_F & 0_F \\ \vdots & \ddots & \ddots & a_{ij} & \vdots \\ 0_F & 0_F & 0_F & 1_F & 0_F \\ 0_F & \dots & \dots & 0_F & 1_F \end{pmatrix} \in U_{n+2}(F) \right\}$$

(where  $a_{ij}$  is in the  $ij$  entry) is relatively minimal in  $U_{n+2}(F)$ .

*Proof.* We prove the assertion for two cases: First case:  $i = 1$  or  $j = n+2$  (that is the indexes from the first row or from the last column) and the second case:  $i > 1$ ,  $j < n+2$ . Let us consider the first case: we know by Remark 3.7 that the set  $S$  of square matrices of size  $n+2$ :

$$B = \begin{pmatrix} 1_F & x_1 & x_2 & \dots & x_{n-1} & x_n & 0_F \\ 0_F & 1_F & 0_F & 0_F & 0_F & 0_F & y_1 \\ 0_F & 0_F & \ddots & \ddots & \ddots & \vdots & y_2 \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0_F & \vdots \\ \vdots & \vdots & \ddots & \ddots & 1_F & 0_F & y_{n-1} \\ 0_F & 0_F & \ddots & \ddots & 0_F & 1_F & y_n \\ 0_F & 0_F & 0_F & \dots & \dots & 0_F & 1_F \end{pmatrix}$$

is relatively minimal in  $H(F^n)$ . Since  $H(F^n)$  is a subgroup of  $U_{n+2}(F)$  we get by Lemma 3.8 that  $S$  is relatively minimal in  $U_{n+2}(F)$ . Now,  $G_{1j}^{n+2}(F) \subset S$  for every  $1 < j < n+2$  and  $G_{in+2}^{n+2}(F) \subset S$  for every  $1 < i < n+2$ . By Lemma 3.8 we obtain that  $G_{ij}^{n+2}(F)$  is relatively minimal in  $U_{n+2}(F)$  for every pair of indexes  $(i, j)$  such that  $i = 1$  or  $j = n+2$  (in addition to the demands:  $i < j$  and  $(i, j) \neq (1, n+2)$ ).

Case 2:  $i > 1$ ,  $j < n+2$ . Fix  $n \in \mathbb{N}$  and a pair  $(i, j)$  such that  $1 < i < j < n+2$ . We shall show that  $G_{ij}^{n+2}(F)$  is relatively minimal in  $U_{n+2}(F)$ . We define the following subgroup of  $U_{n+2}(F)$ :

$$\tilde{U}_{n+2}(F) := \{A \in U_{n+2}(F) \mid a_{kl} = 0_F \text{ if } l \neq k < i\}$$

(it means that the first  $i-1$  rows of every matrix contain only  $0_F$  at each entry (besides the diagonal)). Clearly, this group is isomorphic to the group  $U_{(n+2-(i-1))}(F) =$



$U_{n+3-i}(F)$ . Indeed, for every matrix  $A \in \tilde{U}_{n+2}(F)$  if we delete the first  $i - 1$  rows and the first  $i - 1$  columns we obtain a matrix which belongs to  $U_{n+3-i}(F)$  and it also clear that this way we obtain an isomorphism. Denote this isomorphism by  $f$ . Now,  $G_{ij}^{n+2}(F)$  is a subgroup of  $\tilde{U}_{n+2}(F)$  and  $f(G_{ij}^{n+2}(F)) = G_{1j+1-i}^{n+3-i}(F)$ . Since  $1 < i < j < n + 2$  we obtain that  $i \leq n$  and hence  $n + 3 - i \geq 3$ . Therefore, we can use the reduction to case (1) to obtain that  $G_{1j+1-i}^{n+3-i}(F)$  is relatively minimal in  $U_{n+3-i}(F)$ . By applying Lemma 3.8 (with  $G_1 := U_{n+3-i}(F)$ ,  $G_2 := \tilde{U}_{n+2}(F)$ ,  $H_1 := G_{1j+1-i}^{n+3-i}(F)$  and  $H_2 := G_{ij}^{n+2}(F)$ ) we can conclude that  $G_{ij}^{n+2}(F)$  is relatively minimal in  $\tilde{U}_{n+2}(F)$  and hence also in  $U_{n+2}(F)$  which contains  $\tilde{U}_{n+2}(F)$  as a subgroup. This completes our proof.  $\square$

**Remark 3.10** *In the particular case of  $F = \mathbb{R}$  we obtain by previous results that for every  $n \in \mathbb{N}$  each of the subgroups  $G_{ij}^{n+2}(\mathbb{R})$  is relatively minimal in  $SL_{n+2}(\mathbb{R})$ . It is derived from the fact that  $SL_m(\mathbb{R})$  is minimal for every  $m \in \mathbb{N}$  (see [8, 1]). These groups are also relatively minimal in  $GL_{n+2}(\mathbb{R})$  which contains  $SL_{n+2}(\mathbb{R})$  as a subgroup (see Lemma 3.8). Nevertheless, the fact that these groups are relatively minimal in  $U_{n+2}(\mathbb{R})$  is not derived from the minimality of  $SL_{n+2}(\mathbb{R})$  since  $U_{n+2}(\mathbb{R})$  is contained in  $SL_{n+2}(\mathbb{R})$  and not the opposite (that is  $SL_{n+2}(\mathbb{R})$  is not a subset of  $U_{n+2}(\mathbb{R})$ ).*

**Definition 3.11** *Let  $K$  be a Hausdorff topological division ring. A topological  $K$ -vector space  $E$  is straight if  $E$  is Hausdorff and for every nonzero  $c \in E$ ,  $\lambda \rightarrow \lambda c$  is a homeomorphism from  $K$  to the one-dimensional subspace  $Kc$  of  $E$ . The Hausdorff topological division ring is straight if every Hausdorff  $K$ -vector space is straight.*

**Theorem 3.12** [10, Theorem 13.8] *A nondiscrete locally retrobounded division ring is straight. In particular, a division ring topologized by a proper absolute value is straight.*

**Lemma 3.13** *Let  $(F, \tau)$  be a unital Hausdorff topological ring. Consider the following cases:*

1.  $(F, \tau)$  is a minimal topological group.
2. The multiplication map  $w : (F, \tau) \times (F, \tau) \rightarrow (F, \tau)$  is strongly minimal.
3.  $(F, \tau)$  is minimal as a topological module over  $(F, \tau)$  (i.e. there is no strictly coarser Hausdorff topology  $\sigma$  on  $F$  for which  $(F, \sigma)$  is a topological module over  $(F, \tau)$ ).
4.  $(F, \tau)$  is minimal as a topological ring (i.e. there is no strictly coarser Hausdorff ring topology on  $F$ ).

Then:

$$(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4).$$

*Proof.* (1)  $\Rightarrow$  (2): If  $F$  is a unital topological ring then  $w$  is minimal. Indeed, let  $(\sigma_1, \tau_1, \nu_1)$  be a compatible triple then the identity maps  $(F, \sigma_1) \rightarrow (F, \tau)$  and  $(F, \tau_1) \rightarrow (F, \tau)$  are continuous since the multiplication map  $w : (F, \sigma_1) \times (F, \tau_1) \rightarrow (F, \tau)$  is continuous at  $(\lambda, 1_F), (1_F, \lambda)$  for every  $\lambda \in F$  and from the fact that

$$\forall \lambda \in F \quad w(\lambda, 1_F) = w(1_F, \lambda) = \lambda.$$

Clearly, in the case of a minimal topological Hausdorff group the definition of a minimal biadditive mapping and a strongly minimal biadditive mapping coincide. The rest of the implications are trivial.  $\square$

**Remark 3.14** Although (1)  $\Rightarrow$  (2), the converse implication in general is not true. For instance, the multiplication map  $w : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  is strongly minimal but  $\mathbb{R}$  is not minimal as a topological Hausdorff group.

**Lemma 3.15** Let  $(R, \tau)$  be a straight division ring. Let  $\tau_0$  be a strictly coarser Hausdorff topology on  $\tau$ . Then  $(R, \tau_0)$  is not a topological vector space over  $(R, \tau)$ .

*Proof.* Let  $\tau_0 \subset \tau$ . We shall show that if  $(R, \tau_0)$  is a topological vector space then  $\tau_0 = \tau$ . In the definition of straight division ring let  $K = (R, \tau)$  and  $E = (R, \tau_0)$  also let  $c = 1$ . Then it is clear that the identity mapping  $(R, \tau) \rightarrow (R, \tau_0)$  is a homeomorphism. Hence,  $\tau = \tau_0$ .  $\square$

**Remark 3.16** By our new results we get that in the case of archimedean absolute value, conditions (2)-(4) of Lemma 3.13 hold. Since a proper non-archimedean absolute valued division ring is a straight division ring we get by Lemma 3.15 that the conditions (3)-(4) in Lemma 3.13 hold in this situation. The question that remains open is whether the multiplication map

$$w : (F, \tau) \times (F, \tau) \rightarrow (F, \tau)$$

is strongly minimal where  $F$  is a division ring and the topology  $\tau$  is induced by a proper non-archimedean absolute value. We ask even more concretely: is the multiplication map

$$w : \mathbb{Q} \times \mathbb{Q} \rightarrow \mathbb{Q}$$

strongly minimal when  $\mathbb{Q}$  is equipped with the  $p$ -adic topology?

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